

Implicatures of modified numerals: quality or quantity?¹

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Abstract. This paper presents a theory of modified numerals that derives a three-way distinction in the implicature profile between superlative modifiers, comparative modifiers, and bare numerals. In contrast to the recent proposal by Schwarz (2016b), and drawing on elements from Coppock and Brochhagen’s (2013) inquisitive analysis, the proposal decouples ignorance implicatures from upper-bounding implicatures, and thereby captures an important difference between *more than* and *at least*, which differ in their ignorance implicatures but both lack an upper-bounding implicature. At the same time, it accounts for the context-sensitivity in the ignorance implicatures of modified numerals found by Westera and Brasoveanu (2014), and addresses a problem with Coppock and Brochhagen (2013) pointed out by Schwarz (2016a). The key feature of the proposal is the fact that ignorance implicatures may arise in two different ways, namely, both from the Maxim of Quantity and from the Maxim of Quality.

Keywords: modified numerals, ignorance implicatures, inquisitive semantics.

1. Introduction

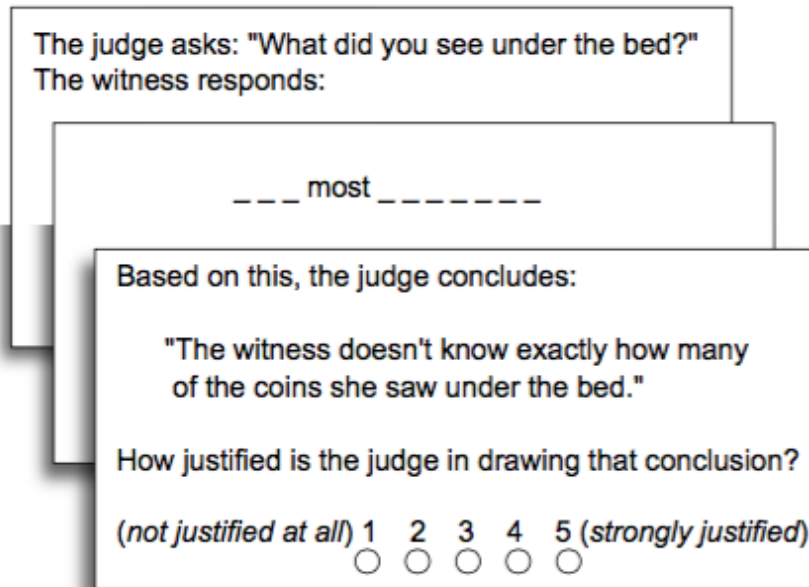
1.1. Empirical targets

We will be concerned with three types of modified numerals: *at least n*, *more than n*, and *n or more*. Many authors have observed that these contrast with each other, as well as with bare numerals, in the implicatures that they give rise to. The basic empirical picture, which is assumed in most work on the topic, is as follows (where the ignorance implicature of *at least six* and *six or more* is not just that the speaker does not know exactly how many sides a hexagon has, but also that she considers it possible that it has precisely six sides).

- | | | | | |
|-----|----|-------------------------------------|------------------------------------|----------------------------------|
| (1) | a. | A hexagon has six sides. | \rightsquigarrow exactly six | $\not\rightsquigarrow$ ignorance |
| | b. | A hexagon has more than five sides. | $\not\rightsquigarrow$ exactly six | $\not\rightsquigarrow$ ignorance |
| | c. | A hexagon has at least six sides. | $\not\rightsquigarrow$ exactly six | \rightsquigarrow ignorance |
| | d. | A hexagon has six or more sides. | $\not\rightsquigarrow$ exactly six | \rightsquigarrow ignorance |

Westera and Brasoveanu (2014) argue based on experimental data that this basic empirical picture is actually a bit too simplistic. They presented experimental participants with a courtroom dialogue, in which a judge asks the witness a question (e.g. *What did you see under the bed?*) and the witness responds with a sentence containing a modified numeral, e.g. *I saw at most 10*

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- POLAR Did you find {at most / less than} ten of the diamonds under the bed?
- WHAT What did you find under the bed?
- HOWMANY How many of the diamonds did you find under the bed?
- APPROX Approximately how many of the diamonds did you find under the bed?
- EXACT Exactly how many of the diamonds did you find under the bed?
- DISJUNCT Did you find eight, nine, ten, or eleven of the diamonds under the bed?

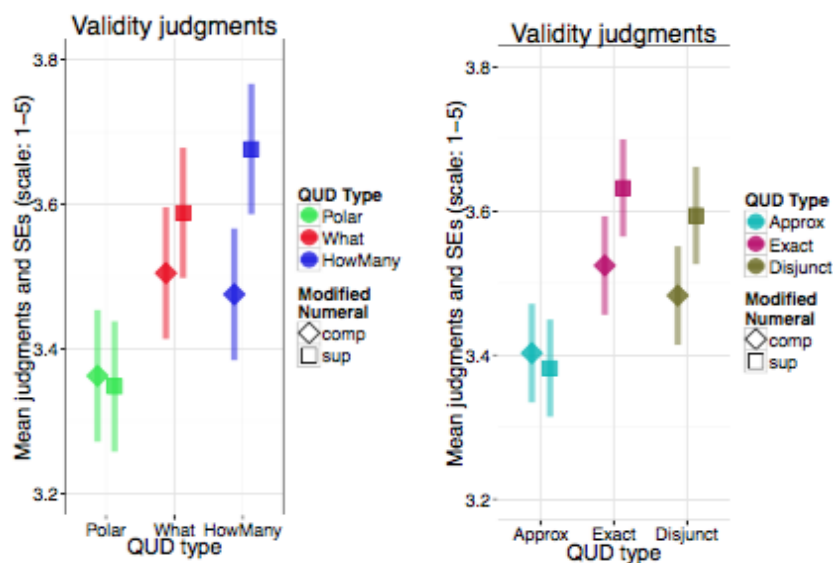


Figure 1: Westera and Brasoveanu's (2014) design and results.

diamonds under the bed. The type of question was experimentally manipulated as indicated in Figure 1 below, and the witness's response always contained either *at most 10* or *less than 10*. The participant is then told that the judge concludes that the witness does not know exactly how many of the relevant kind of items she saw under the bed (the *ignorance inference*), and asked how justified the judge is in drawing that conclusion, on a 1-5 scale.

Their results (see Figure 1) show that comparative modifiers can signal ignorance (e.g. in response to 'how many' questions), and that ignorance can disappear for superlative modifiers (in response to certain polar questions). Note, however, that in most contexts, in particular in response to 'how many' questions, superlative modifiers do give rise to stronger ignorance implicatures than comparative modifiers, in line with what had been assumed in the literature. Another point to notice is that W&B's 'polar question' context involves an echo response.

- (2) A: Did you find at most 10 of the diamonds under the bed?
B: I found at most 10 of the diamonds under the bed.

This may be essential for the ignorance implicature not to arise. Compare:

- (3) A: Did Johnny eat at least four apples today?
B: Yes, he ate at least four apples. \nearrow ignorance

- (4) A: Did Johnny eat more than three apples today?
B: Yes, he ate at least four apples. \rightsquigarrow ignorance

If we compare *more than* and *at least* in non-echo responses to a polar question, the latter seems to implicate ignorance but the former doesn't:

- (5) Context: *Johnny's diet prescribes that he eat at most three apples per day.*
A: Did Johnny stick to his diet today?
B: No, he ate more than three apples. \nearrow ignorance
B': No, he ate at least four apples. \rightsquigarrow ignorance

Moreover, 'out of context', *at least* signals ignorance as well, unlike *more than*:

- (6) a. I grew up with more than two parents.
b. ??I grew up with at least three parents.

Finally, ignorance implicatures triggered by *at least*, unlike implicatures triggered by *more than*, so not seem to be cancelable.

- (7) a. He has more than 10 cars. In fact, he has 12.
b. He has at least 10 cars. #In fact, he has 12.

So, while we concede that *more than* can trigger an ignorance implicature in response to a 'how many' question, and that the ignorance implicature for *at least* can disappear in the context

of certain polar question-answer scenarios (of the echo variety), it seems that the ignorance implicature triggered by *at least* arises more widely than the one triggered by *more than*, and it is of a more obligatory and more robust nature.² What is the source of this difference?

1.2. Quality or quantity?

At least two approaches have been explored in the literature to explain the ignorance implicatures for superlative modifiers. One approach (e.g., [Mayr, 2013b](#); [Kennedy, 2015](#); [Schwarz, 2016b](#)) is to derive ignorance from a particular way of computing **quantity** implicatures. Differences between the various kinds of bare/modified numerals are accounted for on this approach by assuming that they activate different pragmatic alternatives.

Another approach ([Coppock and Brochhagen, 2013](#)) is to derive ignorance as a **quality** implicature. The standard Gricean quality maxim, however, does not suffice for this purpose. Rather, [Coppock and Brochhagen](#) invoke a quality maxim that is not only concerned with the *informative* content of the uttered sentence, but also with its *inquisitive* content, i.e., the semantic alternatives that it introduces. Differences between the various kinds of bare/modified numerals are accounted for on this approach by assuming that they introduce different semantic alternatives.

Note that in other empirical domains (e.g., free choice effects of disjunction under modals or in the antecedent of a conditional), these two approaches have also both been pursued. We will suggest that, in the domain of modified numerals, a **combination** of the two approaches is in fact needed. We will develop such a combined account, and show that it improves on earlier proposals which placed the entire explanatory burden either on quantity or on quality.

2. Previous approaches

2.1. Quantity-based

We will first review a specific quantity-based account of ignorance implicatures, reformulating it in a way that will allow for easy comparison with our own approach. We focus on the proposal of [Schwarz \(2016b\)](#), but see [Mayr \(2013a\)](#) and [Kennedy \(2015\)](#) for closely related proposals.

Schwarz is concerned with *at least n* (not with *more than n* or with *n or more*). He assumes that $\langle \textit{at least, only} \rangle$ forms a Horn scale, along with $\langle 1, 2, 3, \dots \rangle$. This yields the following set of pragmatic alternatives for *Al hired at least two cooks*:

²Shortly before the deadline for submitting the present paper, we became aware of the work of [Mayr and Meyer \(2014\)](#), who make very similar empirical observations, and offer an alternative account of the crucial datapoints. Detailed comparison with the present approach must be left for a future occasion.

[1]	only 1	[1 2 3 4 ...)	at least 1
[2]	only 2	[2 3 4 ...)	at least 2
[3]	only 3	[3 4 ...)	at least 3
[3]	only 4	[4 ...)	at least 4
...

The meanings are visually represented in a way that brings out the fact that none of them are *innocently excludable*, as we will explain below.

To articulate the pragmatics, we introduce the following background notions and notation:

- A speaker's **information state** is a non-empty set of worlds.
- A state s **supports** a sentence φ iff $s \subseteq \llbracket \varphi \rrbracket$.
- A state s **rejects** a sentence φ iff $s \cap \llbracket \varphi \rrbracket = \emptyset$.
- We use A_φ to denote the set of **lexically determined pragmatic alternatives** for φ .

Implicatures can be seen as imposing constraints on what the speaker's information state might be. On Schwarz's approach, they are derived using the following recipe, based on Innocent Exclusion. Start with the **quality implicature** that the speaker's state s supports φ :

$$0_\varphi := \{s \mid s \text{ supports } \varphi\}$$

Now derive **primary quantity implicatures**: the speaker's state does not support any alternative $\psi \in A_\varphi$ that is stronger than φ itself. Let A_φ^\subset be the set of those stronger alternatives:

$$A_\varphi^\subset := \{\psi \in A_\varphi \mid \llbracket \psi \rrbracket \subset \llbracket \varphi \rrbracket\}$$

$$1_\varphi := \{s \in 0_\varphi \mid s \text{ does not support any } \psi \in A_\varphi^\subset\}$$

Now derive **secondary quantity implicatures** by identifying all alternatives $\psi \in A_\varphi^\subset$ satisfying the following two conditions:

1. ψ is **not known** by the speaker according to 1_φ . That is, no $s \in 1_\varphi$ supports ψ .
2. ψ is **innocently excludable** relative to φ (Gazdar, 1979; Fox, 2007).

In a nutshell, ψ is innocently excludable if, whenever a set of alternatives in A_φ^\subset has been consistently rejected we can always go on to reject ψ in addition, maintaining consistency. More precisely: for every subset A' of A_φ^\subset , if there are information states that validate the quality implicature and primary quantity implicatures of φ while rejecting every sentence in A' , i.e., if:

$\{s \in 1_\varphi \mid s \text{ rejects every sentence in } A'\}$ is **non-empty**

then there are also information states that validate the quality implicature and primary quantity implicatures of φ while rejecting every sentence in A' *as well as* ψ :

$\{s \in 1_\varphi \mid s \text{ rejects every sentence in } A' \cup \{\psi\}\}$ is **non-empty**

If $\psi \in A_\varphi^c$ is not known by the speaker according to 1_φ and innocently excludable relative to φ , then we say that ψ is **eligible for a secondary quantity implicature**.

$2_\varphi := \{s \in 1_\varphi \mid s \text{ rejects every } \psi \in A_\varphi^c \text{ eligible for a secondary quantity implicature}\}$

Uttering a sentence φ against the background of a question Q in information state s is **licensed** only if $s \in 0_\varphi$ (speaker adheres to Quality) and $s \in 1_{\varphi,Q} \cap 2_{\varphi,Q}$ (speaker adheres to Quantity). Since $0_\varphi \subseteq 1_{\varphi,Q} \subseteq 2_{\varphi,Q}$, this amounts to saying that $s \in 2_{\varphi,Q}$.

None of the pragmatic alternatives that Schwarz assumes for *Al hired at least two cooks* (see above) is innocently excludable. For instance, rejecting ‘only two’ is not consistent with rejecting ‘at least three’, given the quality assumption that ‘at least two’. So we get primary quantity implicatures, but no secondary ones. Hence ignorance is derived, and no ‘upper bounding’ implicature (exactly n) arises, as desired.

Shortcomings This approach entails a very tight coupling between ignorance implicatures and upper bounding implicatures. A consequence of this is that it is unclear how to distinguish *more than* from *at least*; as mentioned above, both lack upper bounding implicatures, but behave differently with respect to ignorance.

Furthermore, the effects of the QUD documented by [Westera and Brasoveanu \(2014\)](#) are not immediately accounted for (although Schwarz makes it clear that the theory should ultimately be refined, restricting the set of pragmatic alternatives to those that are contextually relevant). We aim to remedy both of these shortcomings in the proposal below.

2.2. Quality-based

The quality-based approach that we will build on ([Coppock and Brochhagen, 2013](#)) is formulated in inquisitive semantics ([Ciardelli et al., 2013](#)). In this framework, every sentence generates a set of semantic alternatives (where each semantic alternative is a set of possible worlds). If a sentence generates two or more alternatives, it is thought of as expressing an issue as to which of these alternatives holds.

[Coppock and Brochhagen](#) propose that the set of alternatives generated by an *at least* sentence consists of all those alternatives that are pragmatically at least as strong as one of the alternatives

generated by the prejacent. For example, *At least two apples fell* generates the set of alternatives corresponding to *Two apples fell*, *Three apples fell*, *Four apples fell*, etc. Assuming a one-sided analysis of bare numerals, this amounts to the following:

(8) *At least two apples fell*: $\{[2,\dots], [3,\dots], [4,\dots], \dots\}$

On a two-sided analysis, the denotation would be:

(9) *At least two apples fell*: $\{[2], [3], [4], \dots\}$

Coppock and Brochhagen further assume, besides the standard Gricean quality maxim, ‘Don’t claim things you don’t believe to be true’, an inquisitive quality maxim as well, which can be characterized informally as: ‘Don’t utter an inquisitive sentence if you already know how to resolve the issue that it expresses’ (cf. Groenendijk and Roelofsen, 2009). More technically, if a sentence generates multiple alternatives, then, when restricted to the speaker’s state, it should still generate multiple alternatives. This, together with the semantics in (8) or (9), derives ignorance implicatures for *at least* sentences.

Shortcomings C&B capture the fact that *at least* generates ignorance implicatures but no upper bounding implicatures, and the fact that bare numerals exhibit exactly the opposite pattern. The analysis is also QUD-sensitive. However, for *more than* they predict that ignorance implicatures do not arise at all, which is in conflict with Westera and Brasoveanu’s (2014) experimental results. Moreover, they do not derive the lack of upper bounding implicatures for *more than*. This is a symptom of a deeper problem, which C&B share with the quantity-based approach: the coupling between ignorance implicatures and the lack of upper bounding implicatures is too tight. This means that it becomes difficult, if not impossible, to capture the differences and similarities between *more than* and *at least*: they behave differently with respect to ignorance, but they both lack upper bounding implicatures.

Even if we focus just on *at least*, the C&B account is not fully satisfactory, because, as pointed out by Schwarz (2016a), the ignorance implicature that is derived for *at least n* is too weak: the approach predicts that a speaker uttering (9) should not know exactly how many apples fell, but not that she should consider it possible that exactly *n* apples fell.

Finally, there is framework issue: C&B formulate their account in ‘unrestricted’ inquisitive semantics, Inq_U , an extension of the standard, basic inquisitive semantics framework, Inq_B . Inq_U makes more meanings available than Inq_B does: the latter does not allow for one alternative to be nested in another—as is the case for instance in (8)—while in Inq_U there are no restrictions on alternative-sets (hence the label ‘unrestricted’). However, this extra richness of Inq_U comes at a price. First, the resulting notion of meaning is less well-behaved from a logical point of view. In particular, it does not come with a suitable notion of entailment and therefore lacks the usual algebraic operations on meanings, like *meet* and *join* (cf., Roelofsen, 2013; Ciardelli et al., 2016). Second, Inq_U is arguably also less well-behaved from an empirical point of view: as discussed in detail in Ciardelli and Roelofsen (2016), while Inq_B straightforwardly facili-

tates an uniform redundancy-based account of so-called *Hurford effects* across declaratives and interrogatives (building on [Katzir and Singh, 2013](#), among others), Inq_U seems to render such an account impossible. One question, then, is whether an analysis of modified numerals along the lines of C&B really needs the full expressive power of Inq_U , or whether it could also be formulated in Inq_B .

3. Proposal

We now spell out a hybrid approach, combining insights from the quality- and quantity-based approaches, and overcoming their respective shortcomings. We provide the necessary background notions and notation from Inq_B in Section 3.1, spell out our lexical assumptions in Section 3.2, and then turn to the pragmatic component of the account in Section 3.3.

3.1. Background notions and notation

In Inq_B , the meaning of a sentence φ , denoted $\llbracket \varphi \rrbracket$, is a set of propositions encoding both the information that is conveyed and the issue that is expressed by φ . Namely, φ is taken to convey the information that the actual world is contained in $\text{info}(\varphi) := \bigcup \llbracket \varphi \rrbracket$, and to express an issue which is resolved precisely by those propositions that are in $\llbracket \varphi \rrbracket$. It is assumed that if a proposition p resolves an issue, then any stronger proposition $q \subseteq p$ resolves that issue as well. Thus, $\llbracket \varphi \rrbracket$ is always *downward closed*: if it contains a proposition p it also contains any $q \subseteq p$. Furthermore, it is assumed that the inconsistent proposition, \emptyset , resolves any issue. Thus, $\llbracket \varphi \rrbracket$ always contains \emptyset and is therefore always non-empty. Taken together, sentence meanings in Inq_B are defined as non-empty, downward closed sets of propositions.

In some cases the issue expressed by φ is *trivial*, in the sense that it is already resolved by the information provided by φ itself. This occurs precisely if $\text{info}(\varphi) \in \llbracket \varphi \rrbracket$. A sentence φ is called **inquisitive** just in case the issue it expresses is non-trivial, i.e., just in case $\text{info}(\varphi) \notin \llbracket \varphi \rrbracket$.

Finally, the **alternatives** associated with a sentence φ in Inq_B are those propositions that contain precisely enough information to resolve the issue expressed by φ . Technically, these are the maximal elements of $\llbracket \varphi \rrbracket$:

$$\text{alt}(\varphi) := \{p \in \llbracket \varphi \rrbracket \mid \text{there is no } q \in \llbracket \varphi \rrbracket \text{ such that } p \subseteq q\}$$

Note that, as remarked above, this characterization of alternatives entails that one alternative can never be properly contained in another, otherwise it could not be a maximal element of $\llbracket \varphi \rrbracket$. Also note that if φ is non-inquisitive, it is always associated with a unique alternative, namely $\text{info}(\varphi)$. Vice versa, if φ generates multiple alternatives, then it must be inquisitive.³

³We should note here that there are several perspectives one can take on the connection between inquisitiveness, a semantic notion, and the communicative effects of sentences when uttered in discourse. The perspective assumed here, in the spirit of [Groenendijk \(2009\)](#) and [Coppock and Brochhagen \(2013\)](#), is that even if a sentence is inquisitive, i.e., even if it semantically expresses a non-trivial issue, a speaker who utters this sentence in discourse does not necessarily *raise* this issue. In particular, she does not necessarily *request* a response that addresses the issue. Under this perspective, it is possible to assume that a disjunctive declarative like *John ate*

3.2. Lexical assumptions

Following C&B, we assume that *at least* sentences generate multiple alternatives. However, we adopt a suggestion made by Schwarz (2016a) in his critique of C&B and analyze *at least n* as having the same meaning as *n or more* would have in inquisitive semantics.⁴ For example:

$$(10) \quad \text{alt}(\textit{At least two apples fell}) = \text{alt}(\textit{Two or more apples fell}) = \{ [2], [3, \dots] \}$$

Notice that, unlike in the C&B analysis, the alternatives for *at least n* are not nested within each other; in fact, they are mutually exclusive. This means that the analysis we are proposing can be formulated in the standard inquisitive semantics framework Inq_B , allowing us to avoid the problems that arise in the unrestricted framework Inq_U .⁵

Following C&B we assume that *more than n* contrasts with *at least n* and with *n or more* in that it is associated with a single semantic alternative:⁶

$$(11) \quad \text{alt}(\textit{More than two apples fell}) = \{ [3, \dots] \}$$

Turning now to pragmatic alternatives, we assume that the lexically determined pragmatic alternatives for *at least n* are $\{ \textit{at least } m \mid m \in \mathbb{N} \}$ and $\{ m \mid m \in \mathbb{N} \}$, and similarly for *n or more* and for *more than n*:

- (12) Lexically determined pragmatic alternatives
- at least n*: $\{ \textit{at least } m \mid m \in \mathbb{N} \} \cup \{ m \mid m \in \mathbb{N} \}$
 - n or more*: $\{ m \textit{ or more} \mid m \in \mathbb{N} \} \cup \{ m \mid m \in \mathbb{N} \} \cup \{ \textit{more than } m \mid m \in \mathbb{N} \}$
 - more than n*: $\{ \textit{more than } m \mid m \in \mathbb{N} \} \cup \{ m \mid m \in \mathbb{N} \}$

two or three apples (with falling intonation) is inquisitive, just like the corresponding interrogative *Did John eat two apples, or three?*. One could say that the former is used to make an *assertion* and the latter to ask a *question*, and that in making an assertion, speakers do not raise the issue that the uttered sentence expresses (perhaps their assertion still ‘evokes’ the issue, but the effect is weaker than in the case of a question). Another perspective that one could take (see, e.g., Farkas and Roelofsen, 2016) is that the issue expressed by a sentence is always raised when the sentence is uttered in discourse. Under this perspective, it does not make sense to treat a disjunctive declarative as being inquisitive, on a par with the corresponding disjunctive question. This perspective allows for a more economical interface between semantics and discourse pragmatics, but is not directly compatible with the present proposal.

⁴The idea of treating *at least n* on a par with *n or more* goes back to Buring (2008).

⁵The meaning for *at least n* given here should, of course, be obtained from a general analysis of *at least*, one that allows us to analyze *at least* in combination with arguments other than numerals. Building on Solt (2011) and Coppock (2016), we assume that an expression of the form *at least P* is interpreted relative to a context providing (i) a *comparison class*, which is a set Φ of propositions, including the proposition associated with the prejacent *P* and (ii) a *pragmatic strength ordering*, which is a partial order on Φ , possibly but not necessarily coinciding with entailment. Relative to such a context, *at least P* is associated with two semantic alternatives: one is the exhaustification of the prejacent *P* with respect to the stronger propositions in Φ ; the other is the union of all propositions in Φ which are strictly stronger than *P*. The meaning that we assume in this paper for *at least n* is obtained from this general account by taking the comparison class to be $\Phi = \{ [0], [1], [2], \dots \}$, where the strength ordering corresponds to the usual ordering on natural numbers.

⁶The account of *more than n* given here can be lifted to a general treatment of *more than*, allowing for arguments other than numerals, in a way similar to the one sketched in footnote 5 for *at least*.

In general, we assume that the lexical pragmatic alternatives for an expression φ are obtained either by deleting parts of φ , or by replacing a scalar item in φ with an element of the same scale.⁷ This makes our pragmatic assumptions less stipulative than those of Schwarz (2016b), and more in line with general theories of pragmatic alternatives (see, in particular, Katzir, 2007).

Following Kennedy (2015), we assume that numerals are ambiguous between a one-sided and a two-sided reading, and the choice between these readings is determined by which yields a stronger interpretation. A two-sided meaning is stronger in a simple positive context, so that is what the m alternatives amount to in such a context.

3.3. Pragmatic assumptions

3.3.1. Quality

Following C&B and earlier work on inquisitive pragmatics (Groenendijk and Roelofsen, 2009), we assume that Quality pertains both to the informative and the inquisitive content of the sentence that is uttered. **Informative sincerity** (Gricean Quality) requires that if a speaker utters a sentence φ , her information state s should support the informative content of φ :⁸ $s \subseteq \text{info}(\varphi)$.

On the other hand, **inquisitive sincerity** requires that a speaker should not utter an inquisitive sentence if she already knows how to resolve the issue that the sentence expresses. That is, if φ is inquisitive, then the speaker's information state s should not already resolve the issue expressed by φ :⁹ $s \notin \llbracket \varphi \rrbracket$. Together:

$$(13) \quad s \in \text{sincere}(\varphi) \quad \text{iff} \quad s \subseteq \text{info}(\varphi) \quad \text{and if } \varphi \text{ is inquisitive, then } s \notin \llbracket \varphi \rrbracket$$

3.3.2. Quantity

Following Schwarz (2016b) and many others, we assume that the maxim of quantity is concerned with alternative expressions that the speaker could have used. However, only expressions that are relevant to the question under discussion should be taken into consideration. Thus, unlike Schwarz, we distinguish **lexical** pragmatic alternatives from **contextual** prag-

⁷Strictly speaking, this means that expressions of the form m and *more* also count as alternatives for n or *more*. However, since such expressions are inconsistent, their presence does not affect the predictions of the theory. Therefore, they are left out of consideration and not listed above.

⁸Of course, this maxim and the ones below are only assumed to be in force in a specific type of conversation, namely one in which the participants exchange information in a fully cooperative way.

⁹The original formulation of the inquisitive sincerity maxim in Groenendijk and Roelofsen (2009) makes reference to the common ground: "If a speaker utters a sentence φ that is inquisitive w.r.t. the common ground, then φ should be inquisitive w.r.t. the speaker's information state as well." For our current purposes this qualification is not necessary. Note also that Coppock & Brochhagen operate with a stronger sincerity maxim, which they call the maxim of *interactive sincerity*. On their account this is needed because the predictions that inquisitive sincerity delivers are too weak. On our present account, inquisitive sincerity delivers the right predictions, and interactive sincerity would do so as well; indeed, interactive sincerity boils down to inquisitive sincerity in InqB.

matic alternatives. The set of lexical pragmatic alternatives for a sentence φ is denoted as A_φ . The set of contextual pragmatic alternatives for φ relative to a question under discussion Q , denoted $A_{\varphi,Q}$, contains only those lexical pragmatic alternatives that are **relevant** to Q :

$$(14) \quad A_{\varphi,Q} = \{\psi \in A_\varphi \mid \psi \text{ is relevant to } Q\}$$

What does it mean for ψ to be relevant to Q ? Recall that the semantic alternatives in $\text{alt}(Q)$ are propositions that contain precisely enough information to resolve the issue expressed by Q . They can be thought of, then, as wholly relevant, complete resolutions of Q . Similarly, any union of two or more such alternatives can be thought of as a wholly relevant, partial resolution of Q . Thus, we say that ψ is relevant to Q if and only if $\text{info}(\psi)$ coincides with the union of a set of semantic alternatives in $\text{alt}(Q)$.

For our current purposes we will stay as close as possible to Schwarz's Innocent Exclusion-based recipe for deriving implicatures. The only serious change is that the standard Gricean Quality requirement is replaced by the requirement that the speaker be both informatively and inquisitively sincere; we also restrict attention to relevant alternatives. So the recipe runs as follows:

The first step, as before, is to compute the **quality implicature**:

$$0_\varphi = \{s \mid s \in \text{sincere}(\varphi)\}$$

Next, also as before, we compute **primary quantity implicatures**, based on the assumption that any pragmatic alternative for φ that would have been more informative was apparently not sincerely utterable, either the speaker's information state doesn't support its informative content, or because the speaker can already resolve the issue that it expresses. We restrict the set of pragmatic alternatives here to those that are relevant, $A_{\varphi,Q}$. Let $A_{\varphi,Q}^\subset$ be the set of such alternatives that are stronger than φ itself: $A_{\varphi,Q}^\subset = \{\psi \in A_{\varphi,Q} \mid \text{info}(\psi) \subset \text{info}(\varphi)\}$.

$$1_{\varphi,Q} = \{s \in 0_\varphi \mid \text{for all } \psi \in A_{\varphi,Q}^\subset : s \notin \text{sincere}(\psi)\}$$

Finally, again as before, we compute **secondary quantity implicatures**. The recipe for doing so is the same as on Schwarz's proposal, except that we now take Q into consideration. That is, we identify all pragmatic alternatives ψ in $A_{\varphi,Q}^\subset$ such that:

1. ψ is **not known** by the speaker according to $1_{\varphi,Q}$. That is, no $s \in 1_{\varphi,Q}$ supports $\text{info}(\psi)$.
2. ψ is **innocently excludable** relative to φ and Q .

The second condition is satisfied just in case for every subset A' of $A_{\varphi,Q}^\subset$, if there are information states that validate the quality implicature and primary quantity implicatures of φ while rejecting every sentence in A' , i.e., if:

$\{s \in 1_{\varphi, Q} \mid s \text{ rejects every sentence in } A'\}$ is **non-empty**

then there are also information states that validate the quality implicature and primary quantity implicatures of φ while rejecting every sentence in A' *as well as* ψ :

$\{s \in 1_{\varphi, Q} \mid s \text{ rejects every sentence in } A' \cup \{\psi\}\}$ is **non-empty**

If $\psi \in A_{\varphi, Q}^{\subset}$ is not known by the speaker according to $1_{\varphi, Q}$ and innocently excludable relative to φ and Q , then we say that ψ is **eligible for a secondary quantity implicature**.

$2_{\varphi, Q} = \{s \in 1_{\varphi, Q} \mid s \text{ rejects any } \psi \in A_{\varphi, Q}^{\subset} \text{ eligible for a secondary quantity implicature}\}$

As before, uttering a sentence φ against the background of a question Q in information state s is **licensed** only if $s \in 2_{\varphi, Q}$, that is, only if the speaker adheres to Quality and Quantity.

4. Predictions

We now discuss the predictions that our account makes for sentences involving bare or modified numerals.

4.1. Predictions in the context of a *how many* question

Suppose the question under discussion is (15a), which we take to be associated with the set of alternatives in (15b).

- (15) a. How many apples did John eat?
b. $Q = \{[0], [1], [2], [3], [4], [5], \dots\}$

Bare numerals First let us consider our predictions for the sentence (16a), involving the bare numeral *three*. Again, following Kennedy (2015), we assume that numerals are scopally ambiguous between a one-sided interpretation (e.g. $[3, \dots]$) and a two-sided one (e.g. $[3]$), and that the interpretation that yields the strongest meaning is the one that is chosen. This means that the basic interpretation of the bare numeral example in (16a) is an ‘exactly’ reading:

- (16) a. φ : John ate three apples.
b. $\text{alt}(\varphi) = \{[3]\}$

Since this sentence is not inquisitive, quality simply requires the speaker to believe that the number of apples that John ate is indeed three.

- (17) $0_{\varphi} = \text{sincere}(\varphi) = \{s \mid s \subseteq [3]\}$

It is easy to see that, given this strong quality implicature, quantity implicatures cannot lead to any stronger conclusion about the speaker's state. Thus, for (16a) we predict an exact interpretation, and no ignorance implicature.

Superlative modifiers Next, let us consider the sentence (18a), involving the superlative modifier *at least*. The semantic alternatives for this sentence in our account are given in (18b).

- (18) a. φ : John ate at least three apples.
 b. $\text{alt}(\varphi) = \{[3], [4, \dots]\}$

Consider the quality implicatures that are drawn about the state s of the speaker. As before, sincerity requires that $s \subseteq \text{info}(\varphi)$, that is, $s \subseteq [3, \dots)$. However, since φ is inquisitive, now sincerity also requires $s \not\subseteq \llbracket \varphi \rrbracket$; that is, it requires s not to be included in either of the alternatives for φ ; in other words, the speaker should not believe that the number of apples was exactly three, nor should she believe that the number is larger than three. Formally, we have:

- (19) $0_\varphi = \text{sincere}(\varphi)$
 $= \{s \mid s \subseteq [3, \dots) \text{ and } s \not\subseteq [3] \text{ and } s \not\subseteq [4, \dots)\}$

So, from quality considerations we already infer not only that the speaker believes that the number of apples John ate is at least 3, but also that she considers it possible that this number is exactly 3, and that she considers it possible that it is larger than 3.

Next, consider quantity implicatures. We have assumed that the lexical pragmatic alternatives for φ are sentences of the form $\psi_n = \text{John ate at least } n \text{ apples}$ or of the form $\chi_n = \text{John ate } n \text{ apples}$, for $n \in \mathbb{N}$. All of these sentences are relevant for the question Q , and therefore they qualify as contextual pragmatic alternatives. Thus, $A_{\varphi, Q}^{\subseteq}$ consists of the sentences ψ_n with $n > 3$, as well as χ_n with $n \geq 3$. Primary quantity implicatures require that none of these sentences could be sincerely uttered by the speaker. However, this is already guaranteed by quality. For take any state $s \in 0_\varphi$. Since $s \not\subseteq [4, \dots)$, whenever $n > 3$ we have that $s \not\subseteq \text{sincere}(\psi_n)$ and $s \not\subseteq \text{sincere}(\chi_n)$, because informative sincerity fails for these sentences. Moreover, since $s \not\subseteq [3]$, we also have $s \not\subseteq \text{sincere}(\chi_3)$. What this shows is that $1_{\varphi, Q} = 0_\varphi$, which means that nothing new is concluded by drawing primary quantity implicatures.

Finally, we will show that no contextual alternative in $A_{\varphi, Q}^{\subseteq}$ is eligible for a secondary quantity implicature. Consider for example the 'at least' sentence ψ_5 , *John ate at least 5 apples*. This alternative is not innocently excludable. To see this, consider the set $A' = \{\chi_4\}$, where $\chi_4 = \text{John ate 4 apples}$, which receives an two-sided ('exactly') interpretation in this context because the two-sided interpretation is stronger than the one-sided one. Rejecting χ_4 is consistent with the primary quantity implicatures; some information states in $1_{\varphi, Q}$ reject it (as a witness, take the state $[3] \cup [5]$). But, given the primary and secondary implicatures, rejecting χ_4 is not consistent with rejecting our candidate alternative ψ_5 (*John ate at least 5 apples*); no information state in $1_{\varphi, Q}$ rejects both. In a nutshell, since rejecting ψ_5 forces acceptance of χ_4 , the former is not innocently excludable. Similar reasoning holds for all of the other 'at

least' formulas ψ_n with $n > 3$.¹⁰ Similarly, to show that each bare numeral alternative χ_n is not innocently excludable we can take $A'_n = \emptyset$ if $n = 3$, and $A'_n = \{\chi_m \mid m \geq 4, m \neq n\}$ if $n > 3$.

This shows that no element of $A_{\varphi, Q}^C$ is eligible for a secondary quantity implicature, which means that $2_{\varphi, Q} = 1_{\varphi, Q}$. This means that no secondary quantity implicatures are derived, or more precisely, nothing new is concluded about the state of the speaker by drawing secondary quantity implicatures.

In conclusion, we have $2_{\varphi, Q} = 0_{\varphi} = \{s \mid s \subseteq [3, \dots) \text{ and } s \not\subseteq [3] \text{ and } s \not\subseteq [4, \dots)\}$. Thus, we predict that, from an utterance of (18a) in the context of question (15a), an ignorance implicature is drawn, and no upper bounding implicature. Importantly, the relevant ignorance implicature is not just that the speaker does not know exactly how many apples John ate, but also that the speaker does not know whether John eat exactly three apples or more.

n or more Disjunctions of the form *n or more* are predicted to behave in a parallel fashion to *at least n*. They have the same denotation:

- (20) a. φ : John ate three or more apples.
 b. $\text{alt}(\varphi) = \{[3], [4, \dots)\}$

The lexical alternatives are sentences of the form *m*, *m or more*, and *more than m* for all natural numbers *m*. All of these will be relevant in the context of a 'how many' question, and none of them will be innocently excludable for reasons parallel to the ones just given for *at least*. Thus, once again we will predict a strong ignorance implicature, and no upper bounding implicatures.

Comparative modifiers Finally, consider (21a). We have assumed that this sentence has a unique semantic alternative, given in (21b).

- (21) a. φ : John ate more than two apples.
 b. $\text{alt}(\varphi) = \{[3, \dots)\}$

Let us compute what implicatures are predicted for φ in the context of Q . First consider quality implicatures: since φ is not inquisitive, only the condition $s \subseteq \text{info}(\varphi)$ is relevant to sincerity. Therefore, we have:

- (22) $0_{\varphi} = \text{sincere}(\varphi) = \{s \mid s \subseteq [3, \dots)\}$

So, from quality we infer that the speaker believes that John ate at least three apples.

Now let us turn to quantity implicatures. The set of lexical pragmatic alternatives to φ consists

¹⁰To see this, consider the set $A'_n = \{\chi_4, \dots, \chi_{n-1}\}$ (in particular, take $A'_4 = \emptyset$). Some information states in $1_{\varphi, Q}$ reject all elements of A'_n : if $n = 4$, this holds trivially, as $A'_4 = \emptyset$; if $n > 4$, we can take as a witness the state $[3] \cup [n]$. However, no information state in $1_{\varphi, Q}$ rejects all elements of A'_n in addition to ψ_n .

of all sentences of the form $\psi_n = \text{John ate more than } n \text{ apples}$ or $\chi_n = \text{John ate } n \text{ apples}$ for n a natural number. All of these pragmatic alternatives are relevant to the question Q we are considering, and therefore qualify as contextual pragmatic alternatives. Therefore, $A_{\varphi,Q} = \{\psi_n | n \in \mathbb{N}\} \cup \{\chi_n | n \in \mathbb{N}\}$ and $A_{\varphi,Q}^{\subseteq} = \{\psi_n | n \geq 3\} \cup \{\chi_n | n \geq 3\}$. Just like φ , all pragmatic alternatives ψ_n are non-inquisitive, nor are the pragmatic alternatives χ_n , which means that the sincerity condition boils down to $s \subseteq \text{info}(\psi_n)$ and $s \subseteq \text{info}(\chi_n)$. Therefore, we have the following primary quantity implicatures:

$$(23) \quad \begin{aligned} 1_{\varphi,Q} &= \{s \in 0_{\varphi} \mid s \not\subseteq \text{sincere}(\psi_n) \text{ for } n \geq 3\} \cap \{s \in 0_{\varphi} \mid s \not\subseteq \text{sincere}(\chi_n) \text{ for } n \geq 3\} \\ &= \{s \subseteq [3, \dots) \mid s \not\subseteq [n, \dots) \text{ for } n > 3\} \cap \{s \subseteq [3, \dots) \mid s \not\subseteq [n] \text{ for } n \geq 3\} \\ &= \{s \mid s \subseteq [3, \dots) \text{ and } s \not\subseteq [3] \text{ and } s \not\subseteq [4, \dots)\} \end{aligned}$$

Thus, from the primary quantity implicatures we infer that the speaker does not know the exact number of apples that John ate and, in fact, that she does not know whether John ate exactly three apples or more.

Finally, consider secondary quantity implicatures. Clearly, each pragmatic alternative $\psi \in A_{\varphi,Q}^{\subseteq}$ is not known by the speaker according to $1_{\varphi,Q}$. But none of these alternatives is innocently excludable, for reasons parallel to those given for *at least* above. Thus, we have $2_{\varphi,Q} = 1_{\varphi,Q}$, that is, nothing new is inferred in drawing secondary quantity implicatures.

Summing up, for (21a) in the context of the ‘how many’ question (15a) we predict an ignorance implicature and no upper bounding implicature. While the relevant ignorance inference is exactly the same that was derived above for (18a), there is a crucial difference between the two cases: the inference is derived as a *quantity* implicature for (21a), but as a *quality* implicature for (18a). Quality implicatures are viewed as being of a more fundamental nature than other kinds of implicatures.¹¹ Moreover, they are difficult to cancel (as exhibited by the oddness of Moore sentences) and there is no reason to imagine that they would depend on the question under discussion any more than the content of the utterance does. We take this to account for our observations in Section 1.1, which show that in the case of *at least*, ignorance implicatures arise more robustly and more widely, and are harder to cancel, than in the case of *more than*.^{12,13}

4.2. Predictions in the context of a polar question

Let us now consider a context in which not all of the lexical pragmatic alternatives to our sentences are relevant, and therefore available in the computation of quantity implicatures.

¹¹For instance, Grice (1975: p.27) writes: “It is obvious that the observance of some of these maxims is a matter of less urgency than is the observance of others; a man who has expressed himself with undue prolixity would, in general, be open to milder comment than would a man who has said something he believes to be false”.

¹²Lauer (2014) argues that the ignorance implicatures generated by disjunctions are also of a mandatory nature.

¹³Interestingly, note that in the case of (18a), even if an addressee does not derive the inference as a quality implicature (i.e., if she takes it for granted that the speaker adheres to informative sincerity, but not necessarily to inquisitive sincerity as well), then she would *still* derive ignorance as a quantity implicature, in a way parallel to what we discussed for (21a). This, we suggest, lends additional robustness to the ignorance implicature of *at least* in comparison to that of *more than*.

Suppose that John’s diet prescribes that he eat at most two apples per day, and does not prescribe anything else. Consider the polar question in (24a). Given our contextual assumptions, the alternatives for this question are the ones displayed in (24b).

- (24) a. Did John stick to his diet today?
b. $Q = \{[0, 2], [3, \dots]\}$

Compare the following responses to (24a):

- (25) a. No, he ate more than two apples.
b. No, he ate at least three apples.

Intuitively, upon hearing the response (25a) we do not conclude that the speaker is ignorant as to the number of apples that John ate, whereas we do so upon hearing (25b). In other words, the ignorance implicature associated with (25b) still arises in this context, but the implicature stemming from (25a) does not.

This is indeed predicted on our account. To see why, first consider (25b): we have seen above that for this sentence, the ignorance inference arises as a quality implicature; since quality implicatures are context-independent on our account, this implicature is still predicted in the present setting. The same holds for *n or more*.

The situation is different for (25a). For this sentence, an ignorance inference in the context of question (15a) was derived as a quantity implicature. However, on our account the computation of quantity implicatures is sensitive to the question Q under discussion: only lexical pragmatic alternatives which are relevant to Q are taken into account. In the present context, no lexical alternative to φ —except for the sentence φ itself—is relevant to Q . Thus, $A_{\varphi, Q}^C = \emptyset$, which means that no primary or secondary quantity implicatures are derived. Thus, we predict that the ignorance implicature that we found above for (25a) disappears in the context of the polar question (24a).¹⁴

5. Conclusion

The proposal we have made here allows us to overcome the shortcomings of previous accounts. It achieves a three-way contrast between superlative modifiers, comparative modifiers, and numerals, in contrast to Schwarz (2016b). It furthermore accounts for the QUD-sensitivity observed by Westera and Brasoveanu (in contrast to both Schwarz and C&B). It predicts ignorance with respect to the prejacent of *at least* (overcoming Schwarz’s critique of C&B). And it brings C&B’s approach in line with recent theorizing on inquisitive semantics, where one alternative can never entail another.

Most importantly, we account for the following facts: *more than* indeed can imply ignorance in *how many* contexts, as observed by Westera & Brasoveanu, but the ignorance implicature

¹⁴In addition, notice that, due to the absence of contextually relevant alternatives, we still correctly predict the lack of upper bounding implicatures for both (25a) and (25b).

of *at least* is more robust, as witnessed by (i) the fact that it is perceived to be stronger than the ignorance implicature of *more than* in *how many* contexts; (ii) the fact that it persists in non-echoic responses to polar questions; and (iii) the fact that it contrasts with *more than* ‘out of context’, as in the following example, repeated from the introduction:

- (26) a. I grew up with more than two parents.
b. ??I grew up with at least two parents.

To obtain these results, it is crucial to be able to derive ignorance implicatures through two distinct routes: quality for *at least*, and quantity for *more than*.

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